1. Let $\varepsilon > 0$, $\exists\ \delta > 0$ such that $\Big|\,|(x'(u_1), y'(v_1))| - |(x'(u_2), y'(v_2))|\,\Big| < \dfrac{\varepsilon}{b-a}$

if $|u_1 - u_2| < \delta$, $|v_1 - v_2| < \delta$

Let P be partition of $[a,b]$ such that $|P| < \delta$

By Mean value theorems,

$$
\begin{aligned}
|\alpha(t_i) - \alpha(t_{i-1})| &= |(x(t_i) - x(t_{i-1}), y(t_i) - y(t_{i-1}))| \\
&= |(x'(c_{i1})(t_i - t_{i-1}), y'(c_{i2})(t_i - t_{i-1}))| \\
&= |(x'(c_{i1}), y'(c_{i2}))|(t_i - t_{i-1})
\end{aligned}
$$

$\int_{t_{i-1}}^{t_i} |\alpha'(t)|\,dt = |\alpha'(u_i)|(t_i - t_{i-1})$ where $c_{i1}, c_{i2}, u_i \in (t_{i-1}, t_i)$

Then

$$
\begin{aligned}
&\left|\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| - \int_a^b |\alpha'(t)|\,dt\right| \\
&= \left|\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| - \int_{t_{i-1}}^{t_i} |\alpha'(t)|\,dt\right| \\
&= \left|\sum_{i=1}^{n} \big(|(x'(c_{i1}), y'(c_{i2}))| - |\alpha'(u_i)|\big)(t_i - t_{i-1})\right| \\
&\leq \sum_{i=1}^{n} \Big|\,|(x'(c_{i1}), y'(c_{i2}))| - |\alpha'(u_i)|\,\Big|(t_i - t_{i-1}) \\
&< \sum_{i=1}^{n} \frac{\varepsilon}{b-a}(t_i - t_{i-1}) = \varepsilon
\end{aligned}
$$

Since $\int_{t_{i-1}}^{t_i} |\alpha'(t)|\,dt \geq |\alpha(t_i) - \alpha(t_{i-1})|$ $\quad i = 1, 2, \ldots, n$

$$
\begin{aligned}
\int_a^b |\alpha'(t)|\,dt &= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |\alpha'(t)|\,dt \\
&\geq \sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| = L_a^b(\alpha, P)
\end{aligned}
$$

$\therefore \int_a^b |\alpha'(t)|\,dt \geq \sup\{L_a^b(\alpha, P) \mid P \text{ is a partition of } [a,b]\}$

By above result, let $\varepsilon > 0$, $\exists$ ~~[illegible]~~ partition P of $[a,b]$ with $|P| < \delta$ such that

$$\left|L_a^b(\alpha, P) - \int_a^b |\alpha'(t)|\,dt\right| < \varepsilon$$

Then $\int_a^b |\alpha'(t)|\, dt < L_a^b(\alpha, P) + \varepsilon$

$\leq \sup\{L_a^b(\alpha,P) \mid P \text{ is a partition of } [a,b]\} + \varepsilon$

By letting $\varepsilon \longrightarrow 0$,

$$\int_a^b |\alpha'(t)|\, dt \leq \sup\{L_a^b(\alpha,P) \mid P \text{ is a partition of } [a,b]\}$$

$\therefore \int_a^b |\alpha'(t)|\, dt = \sup\{L_a^b(\alpha,P) \mid P \text{ is a partition of } [a,b]\}$

2. If $|\alpha(t)|$ is nonzero constant, then

$\langle \alpha(t), \alpha(t) \rangle = c$ for some constant $c > 0$

$\frac{d}{dt} \langle \alpha(t), \alpha(t) \rangle = 0$

$\langle \alpha(t), \alpha'(t) \rangle = 0$

If $\langle \alpha(t), \alpha'(t) \rangle = 0$, then $\frac{d}{dt} |\alpha(t)|^2 = 0$

Hence $|\alpha(t)|^2$ is positive constant

And $|\alpha(t)| \neq 0$ as α is regular curve

3. $\alpha(t) = (3t, 3t^2, 2t^3)$, $\alpha'(t) = (3, 6t, 6t^2)$

$|\alpha'(t)| = 3 + 6t^2$

$$\frac{\alpha'(t)}{|\alpha'(t)|} \cdot (1, 0, 1) = \frac{3+6t^2}{3+6t^2} = 1 \quad \text{which is constant}$$

4. At time t, centre of the disk : $(t, 1)$

angle of the point on the disk : $\theta(t) = \pi + t$

$\alpha(t) = (t, 1) + (\sin(\theta(t)), \cos(\theta(t)))$

$= (t - \sin t, 1 - \cos t)$

$\alpha'(t) = (1 - \cos t, \sin t)$

$\alpha'(t) = (0, 0)$ iff $t = 2n\pi$ $n = 0, \pm 1, \pm 2, \ldots$

Arc length of the cycloid corr. to a complete rotation:

$$\int_0^{2\pi} |\alpha'(t)|\,dt = \int_0^{2\pi} \sqrt{2-2\cos t}\,dt$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{t}{2}}\,dt$$

$$= 8$$

5. $\alpha(t) = (\sin t, \cos t + \log\tan\frac{t}{2})$ $\quad t \in (0,\pi)$

$$\alpha'(t) = \left(\cos t, -\sin t + \frac{\sec^2\frac{t}{2}}{\tan\frac{t}{2}} \cdot \frac{1}{2}\right)$$

$$= \left(\cos t, -\sin t + \frac{1}{\sin t}\right) \qquad \alpha'(t) = (0,0) \text{ iff } x = \frac{\pi}{2}$$

The intersection point between the tangent line and y-axis: $\left(0, y(t) - x(t)\frac{y'(t)}{x'(t)}\right)$

The length: $\left|(x(t), y(t)) - \left(0, y(t) - x(t)\frac{y'(t)}{x'(t)}\right)\right|$

$$= \sqrt{x(t)^2 + \left(x(t)\frac{y'(t)}{x'(t)}\right)^2}$$

$$= \sqrt{\sin^2 t + \left(\sin t \cdot \frac{-\sin t + \frac{1}{\sin t}}{\cos t}\right)^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t}$$

$$= 1$$

6. If $\alpha(t) \neq p$; then $\alpha(t) - p \,/\!/\, N(t)$

Claim: $\alpha(t) \neq p \quad \forall t \in I$

Suppose $\alpha(t_0) = p$ for some $t_0 \in I$. ~~By~~ Since α is regular,

$\exists\, \delta > 0$ such that $|\alpha(t_0+h) - \alpha(t_0)| \neq 0 \quad \forall\, 0<h<\delta$

Then $\langle \alpha(t_0+h) - \alpha(t_0), N(t_0+h)\rangle = |\alpha(t_0+h) - \alpha(t_0)| \quad \forall\, 0<h<\delta$

or

$-|\alpha(t_0+h) - \alpha(t_0)| \quad \forall\, 0<h<\delta$

WLOG, we assume $\langle \alpha(t_0+h) - \alpha(t_0), N(t_0+h)\rangle = |\alpha(t_0+h) - \alpha(t_0)| \quad \forall\, 0<h<\delta$

Then $\dfrac{\langle \alpha(t_0+h) - \alpha(t_0), N(t_0+h)\rangle}{h} = \dfrac{|\alpha(t_0+h) - \alpha(t_0)|}{h}$

As $h \to 0$, L.H.S $= 0$, R.H.S $= |\alpha'(t_0)| > 0$. Contradiction arise

Then we can assume $\langle \alpha(t)-p, N(t)\rangle = |\alpha(t)-p| \quad \forall\, t\in I$

$$\frac{d}{dt}\langle \alpha(t)-p, N(t)\rangle = \frac{d}{dt}|\alpha(t)-p|$$

$$\langle \alpha'(t), N(t)\rangle + \langle \alpha(t)-p, N'(t)\rangle = \frac{d}{dt}|\alpha(t)-p|$$

$$0 + 0 = \frac{d}{dt}|\alpha(t)-p|$$

Therefore $|\alpha(t)-p|$ is constant and non zero

So $\alpha(I)$ is contained in a circle of some radius $r>0$ centered at p

7. For any $\alpha(t)\neq p$, we have $\langle T(t), \alpha(t)-p\rangle = \pm|\alpha(t)-p|$ on the ~~each~~ each connected interval of $\{\alpha(t)\neq p\}$

Then $T(t) = \pm\dfrac{\alpha(t)-p}{|\alpha(t)-p|}$

$$\frac{d}{dt}T(t) = \pm\left(\frac{\alpha'(t)}{|\alpha(t)-p|} - \frac{\alpha(t)-p}{|\alpha(t)-p|^2}\frac{\langle \alpha'(t), \alpha(t)-p\rangle}{|\alpha(t)-p|}\right)$$

$$= \pm \frac{\alpha'(t)}{|\alpha(t)-p|} - \frac{\alpha(t)-p}{|\alpha(t)-p|^2}\langle \alpha'(t), T(t)\rangle$$

$$= \pm \frac{\alpha'(t)}{|\alpha(t)-p|} - \frac{\alpha(t)-p}{|\alpha(t)-p|^2}|\alpha'(t)|$$

$$= 0$$

Therefore, $T(t)$ is a fixed vector on each connected interval of $\{\alpha(t)\neq p\}$

Suppose $\alpha(t_0)=p$, then $\exists\, \delta>0$ such that

$$|\alpha(t)-\alpha(t_0)|\neq 0 \quad \forall\ 0<|t-t_0|<\delta \quad \text{as } \alpha \text{ is regular}$$

And $\lim_{t\to t_0^-} T(t) = T(t_0) = \lim_{t\to t_0^+} T(t)$

So $T(t)$ is constant $\forall\, t\in I$

Hence $\alpha(I)$ is contained in a ~~sta~~ straight line passing through p.

Example for α is not regular:

Let $f(t) = \begin{cases} 0 & \text{if } t\leq 0 \\ e^{-\frac{1}{t^2}} & \text{if } t>0 \end{cases}$

$f(t)$ is C^∞ function

$$\alpha(t) = \begin{cases} (0, f(t)) & \text{if } t>0 \\ (0,0) & \text{if } t=0 \\ (f(-t), 0) & \text{if } t<0 \end{cases}$$

$\alpha(t)$ is not regular at $(0,0)$, the trace of α is contained in two straight line